# On the Existence of Monodromies for the Rabi model

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We discuss the existence of monodromies associated with the singular points of the eigenvalue problem for the Rabi model. The complete control of the full monodromy data requires the taming of the Stokes phenomenon associated with the unique irregular singular point. The monodromy data, in particular the composite monodromy, are written in terms of the parameters of the model via the isomonodromy method and the tau-function of the Painlevé V. These data provide a systematic way to obtain the quantized spectrum of the Rabi model.

## INTRODUCTION

The Rabi model [1, 2] describes the interaction of a single electromagnetic mode – a simple harmonic oscillator – with matter – a two-level quantum system. It is a quite simple model yet it has an interesting and rich spectrum. Recently, it has attracted vividly attention due to experimental and mathematical reasons. From applied and experimental physics side, there have emerged interesting applications in quantum optics and quantum computation because of good prospects of experimental realization using Josephson junctions, traped ions and others (see references in [3]). From the mathematical side, for a long time it has been a challenge to prove its exactly solvability. And so, finally, as recently as 2011, Braak [3] solved the model in a Bargmann (coherent state) representation obtaining in a systematic way its regular spectrum as zeroes of a transcendental function. The exceptional part of the spectrum were already known since the late 1970's [4], but it can also be obtained in the framework proposed by Braak [5].

This fact has opened up a whole new set of interesting problems in mathematical physics. One issue regards the notion of integrability of the Rabi model. At face value, the Rabi model could be a first instance of an exactly solvable yet not integrable model in mathematical physics. A controversy ensues since the proper definition of integrability in quantum physics – or even in mathematics – seems not to be clear cut [6, 7]. Braak claimed that the Rabi model is integrable in a new quantum integrability criterion coined by himself. A system is dubbed Braak integrable if there are $f = f_c + f_d$ quantum numbers classifying the eigenstates uniquely, where $f_c, f_d$ stand for the number of continuous and discrete degrees of freedom (d.o.f.), respectively. For the Rabi model, this number is $f_c = f_d = 1$, one harmonic oscillator and one two-level system, so that $f = 2$. These two quantum numbers arises from the $\mathbb{Z}_2$ parity symmetry of the Hamiltonian. Even in the lack of a better mathematical formulation of this criterion – still a pending project – it served as the guiding principle to the solution of this outstanding problem.

Controversies aside, it would be desirable to frame this discussion in a more conservative setup. Therefore, Batchelor and Zhou [7] raised the issue of whether the Rabi model is Yang-Baxter integrable (YBI). They succeed to show YBI for two special points in the space of parameters of the Rabi model. For a generic point in this moduli, YBI is still an open question.

Another important issue arising from the Braak's work is that of finding an universal method that applies to a wide variety of models involving coupling of a boson mode with a two-level system in the Bargmann representation. Such a program has been developed by Maciejewski et al. [8, 9]. They have used a method framed in terms of the Wronskians, that is, a $2 \times 2$ matrix containing both the wave-function and its derivative in the neighbourhood of a singular point.

The present work advances some enlightenment in the direction of proving YBI for a generic point in the moduli of parameters of the Rabi model. The departing point is the Bargmann representation of the Rabi model. In this representation, it can be easily shown [5] that the Rabi model is described by a confluent Heun equation. Here we use the known fact [10, 11] that the monodromy data of these equations can be cast in terms of Painlevé V transcendent tau-function [12] via isomonodromy equations. The global properties, relevant for the problem of the spectrum and for the YBI, are encoded in the notion of composite monodromies. We then present the composite monodromy parameter of the Rabi model. Finally we discuss in general terms how one could use this composite monodromy parameter to obtain the Rabi model's spectrum.

The novelty of our work consists in the presentation of the monodromies associated with the singular points of the ODE arising from the eigenvalue problem of the Rabi model. We then discuss the relevance of the Stokes phe-

nomenon in order to have a complete monodromy data set. We conjecture that it is the emergence of the Stokes phenomenon and the need of extra parameters in the monodromy data set that rendered extra difficulties in the full demonstration of the Yang-Baxter integrability of the Rabi model.

This work is organized as follows: we first write the Rabi model as a standard Fuchsian system, and then discuss the mododromies around the singular points. A special situation happens for the monodromy around the unique irregular singular point, the point at infinity, giving rise to the Stokes phenomenon, which we discuss in detail. As an outcome we obtain the general group relation for the monodromies and how it is related with Yang-Baxter equations. We next discuss the isomonodromy method aiming at writing the composite monodromy parameter in terms of the monodromy parameter at the irregular point and the stokes parameters. On the other hand, the existence of monodromy matrices for our original system are obtained from the tau-function of the Painlevé V. We finally obtain the composite monodromy parameter in terms of the parameters of the Rabi model which provides in a systematic way the quantized spectrum of the Rabi model.

## RABI AND ITS MONODROMIES

The quantum Rabi model (QRM) is described by the Hamiltonian

$$H_R = a^\dagger a + \Delta \sigma^z + g \sigma^x \left( a^\dagger + a \right), \qquad (1)$$

where the boson mode is described by $[a, a^\dagger] = 1$, the fermion mode by the Pauli matrices, $\Delta$ is the level separation of the fermion mode and $g$ is the boson-fermion coupling.

The QRM can be written in terms of two copies of Jaynes-Cummings model (JCM), each with its appropriate chirality. Indeed, the chiral JCM Hamiltonian reads

$$H_{JC} = a^\dagger a + \Delta \sigma^z + g \left( \sigma^+ a + \sigma^- a^\dagger \right), \qquad (2)$$

and the anti-chiral one reads

$$H_{\overline{JC}} = a^\dagger a + \Delta \sigma^z + g \left( \sigma^- a + \sigma^+ a^\dagger \right), \qquad (3)$$

so that

$$H_R = \frac{1}{2} \left( H_{JC} + H_{\overline{JC}} \right) = a^\dagger a + \Delta \sigma^z + g \sigma^x (a^\dagger + a), \quad (4)$$

where we have used $\sigma^x = (\sigma^+ + \sigma^-)/2$. It is important to notice that $[H_{JC}, H_{\overline{JC}}] \neq 0$.

Consider the Ansatz [3, 5]

$$|\psi(a^\dagger)\rangle = f_1(a^\dagger)|0\rangle|+\rangle + f_2(a^\dagger)|0\rangle|-\rangle, \qquad (5)$$

where the harmonic oscillator ground state is defined by $a|0\rangle = 0$, $\sigma^z|\pm\rangle = \pm|\pm\rangle$, and $f_i$, $i = 1, 2$, are analytic functions of $a^\dagger$. We can now use Bargmann's prescription

$$a^\dagger \mapsto w, \qquad a \mapsto \partial_w, \qquad (6)$$

so that $[a, a^\dagger]f(w) = f(w)$. Substituting the Ansatz into the stationary Schroedinger equation, or the eigenvalue equation, $H_R|\psi\rangle = E|\psi\rangle$, we obtain, after setting $f_\pm = f_1 \pm f_2$,

$$\partial_w f_+ = \frac{E - gw}{w + g} f_+ - \frac{\Delta}{w + g} f_-$$
$$\partial_w f_- = -\frac{\Delta}{w - g} f_+ + \frac{E + gw}{w - g} f_-. \qquad (7)$$

Eliminating say $f_+$ in terms of $f_-$, this results in a second order linear differential equation for $f_+$ (and $f_-$) which could be brought to a confluent Heun equation [5]. Let us define

$$z = -2g(w + g), \qquad \Phi(z) = \begin{pmatrix} f_+^{(1)} & f_+^{(2)} \\ f_-^{(1)} & f_-^{(2)} \end{pmatrix}, \qquad (8)$$

where $f_\pm^{(1,2)}$ are the two linearly independent solutions of the system above. The fundamental matrix $\Phi(z)$ is then invertible and unique up to right multiplication of a constant matrix. With this change of variables, we can bring the model to a standard Fuchsian form:

$$\frac{d\Phi}{dz} \Phi^{-1} = \frac{1}{2}\sigma_3 + \frac{1}{z}A_0 + \frac{1}{z - t}A_t, \qquad (9)$$

with $t = -4g^2$ and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} E + g^2 & -\Delta \\ 0 & 0 \end{pmatrix},$$
$$A_t = \begin{pmatrix} 0 & 0 \\ -\Delta & E + g^2 \end{pmatrix}. \qquad (10)$$

The system (9) has two regular singular points at $z_i = 0, t$, $i = 0, t$ and an irregular singular point at $z_\infty = \infty$ with Poincaré index 1. The analytical structure of the system near the regular singular point is characterized by the monodromy matrices $M_0$ and $M_t$, defined as the effect of an analytical continuation around the corresponding singular point,

$$\Phi((z - z_i)e^{2\pi i} + z_i) = \Phi(z)M_i, \qquad i = 0, t. \qquad (11)$$

One notes that, since any two set of solutions $\Phi(z)$ are related by right multiplication, the monodromy matrices are defined up to an overall conjugation. Moreover, one can choose initial conditions of (9) so that, near a regular

singular point $z_i$, one has

$$\Phi(z)|_{z \approx z_i} = \begin{pmatrix} (z - z_i)^{\alpha_i^+} & 0 \\ 0 & (z - z_i)^{\alpha_i^-} \end{pmatrix}. \quad (12)$$

Therefore, one can see that generically the monodromy matrix $M_i$ can be written as

$$M_i = g_i \begin{pmatrix} e^{2\pi i \alpha_i^+} & 0 \\ 0 & e^{2\pi i \alpha_i^-} \end{pmatrix} g_i^{-1}, \quad (13)$$

where $g_i \in \mathrm{SL}(2,\mathbb{C})$ are called the connection matrices. They are also defined up to left multiplication. One also notes that, for algebraic purposes, only the difference $\theta_i = \frac{1}{2}(\alpha_i^+ - \alpha_i^-)$ is important. Overall shift of the coefficients $\alpha_i^\pm$ can be obtained by an s-transformation of the solutions: $f_\pm(z) \to (z - z_i)^a f_\pm(z)$.

The system at the irregular singular point at $z = \infty$ is slightly more complicated, due to the *Stokes phenomenon*. In order to describe it, let us start by noting that close to $z = \infty$ the solutions are of the form $f_\pm^{(1,2)} \approx e^{\pm z/2}$, and the Frobenius series obtained at this point is only formal: its convergence radius is zero. One can see that generically near infinity the system has the form

$$\frac{d\Phi}{dz} \Phi^{-1} = -\frac{1}{2}\sigma_3 + \frac{A_0 + A_t}{z} + \mathcal{O}(z^{-2}). \quad (14)$$

From the coefficient $A_\infty = -(A_0 + A_t)$ of the $z^{-1}$ term one can define the naive monodromy at $z = \infty$, given by the difference of the eigenvalues $\theta_\infty = \alpha_\infty^- - \alpha_\infty^+$. The epithet "naive" comes in because the monodromy structure around $z = \infty$ also depends on the first constant term. To describe this structure, we follow [10, 12, 13] and define the sectors of the complex plane:

$$\mathcal{S}_j = \{z \in \mathbb{C} \,|\, (2j - 5)\frac{\pi}{2} < \arg z < (2j - 1)\frac{\pi}{2}\}, \quad (15)$$

$j = 1, 2, \ldots$. On each $\mathcal{S}_j$ we have the following asymptotic behavior for the solutions of the system (9):

$$\Phi(z)|_j = G_j(z^{-1}) \exp(\tfrac{1}{2}z\sigma_3) z^{-\frac{1}{2}\theta_\infty \sigma_3}, \quad (16)$$

where $G_j(z^{-1}) = \mathbb{1} + \mathcal{O}(z^{-1})$ is analytic near $z = \infty$. The Stokes phenomenon relates the solutions satisfying (16) between different sectors $\mathcal{S}_j$,

$$\Phi_{j+1}(z) = \Phi_j(z) S_j, \quad (17)$$

where $S_k$ are the Stokes matrices. Now, $\Phi_j(e^{2\pi i}z) = \Phi_{j+2}(z)e^{-\pi i\theta_\infty \sigma_3}$ – they are defined in the same domain –, so we have that $S_{j+2} = e^{\pi i\theta_\infty \sigma_3} S_j e^{\pi i\theta_\infty \sigma_3}$. Therefore one can choose a basis where

$$S_{2j} = \begin{pmatrix} 1 & s_{2j} \\ 0 & 1 \end{pmatrix}, \quad S_{2j+1} = \begin{pmatrix} 1 & 0 \\ s_{2j+1} & 1 \end{pmatrix}, \quad (18)$$

where the numbers $s_k$ are called Stokes parameters. By the identification of $S_{j+2}$ with $2\pi$ rotations, one then defines the monodromy at infinity at sector $\mathcal{S}_j$ by

$$M_\infty|_{\mathcal{S}_j} = S_j S_{j+1} e^{i\pi\theta_\infty \sigma_3}, \quad (19)$$

which is the one satisfying the free group relation with the other two monodromy matrices:

$$M_\infty M_t M_0 = \mathbb{1}. \quad (20)$$

Note that once we settle in a sector $\mathcal{S}_j$, say $j = 1$, then knowledge of only two consecutive Stokes parameters $s_1$ and $s_2$ are sufficient to reconstruct the whole series of Stokes matrices $S_j$.

The outcome of the above analysis is that the whole set of parameters $\vec{\theta} = \{\theta_0, \theta_t, \theta_\infty, s_1, s_2\}$ is sufficient to determine the monodromy matrices up to an overall conjugation. This set is thus called the *monodromy data*.

The existence of the monodromy matrices provides an explicit representation of the 3-braid group – in fact, the permutation group $\mathbb{S}_3$ – acting on $M_i$ as

$$\sigma_{ij}(M_i) = M_j M_i M_j^{-1}, \\ \sigma_{ij}(M_j) = (M_j M_i) M_j (M_j M_i)^{-1}. \quad (21)$$

These generators satisfy

$$\sigma_{ij} \circ \sigma_{jk} \circ \sigma_{ki} = \sigma_{ik} \circ \sigma_{kj} \circ \sigma_{ji}, \quad (22)$$

which is known as the Yang-Baxter relation [14]. The existence of the monodromy matrices then assures that the Rabi model is integrable in the algebraic sense.

## THE ISOMONODROMY METHOD

The Riemann-Hilbert problem consists in finding a Fuchsian system with a prescribed set of monodromies. Our problem is the inverse one. In order to solve such inverse Riemann-Hilbert problem, we will first notice that there are many different families of $A_i$'s which give the same monodromy. Some of them are trivially related by overall conjugation. But even so there is still a family of non-trivial set of Fuchsian systems parametrized by the position of an extra singular point $t$.

This family was first described by Schlesinger – see [11, 12, 15, 16] for reviews – but it is more easily understood in terms of flat holomorphic connections [17]. Suppose we set

$$A(z, t) = \frac{1}{2}\sigma_3 + \frac{A_0(t)}{z} + \frac{A_t(t)}{z - t} = \frac{\partial \Phi(z, t)}{\partial t} \Phi^{-1}(z, t) \quad (23)$$

as the "z-component" of a flat connection. It is straightforward to see that, if we consider the "t-component"

as

$$B(z,t) = -\frac{A_t(t)}{z - t}, \qquad (24)$$

then $F = \partial_t A - \partial_z B + [A, B] = 0$ if the $A_i(t)$ satisfy

$$\frac{\partial A_0}{\partial t} = \frac{1}{t}[A_t, A_0],$$
$$\frac{\partial A_t}{\partial t} = -\frac{1}{t}[A_t, A_0] - \frac{1}{2}[A_t, \sigma_3]. \qquad (25)$$

This system are called the *Schlesinger equations*. Since the "field strenght" $F$ vanishes, then the monodromy data of the Fuchsian system (9) will be independent of $t$ if $A_0(t)$ and $A_t(t)$ satisfy the equations (25). The matrices $A_0$ and $A_t$ can be thought of as a Lax pair for the isomonodromy flow.

Despite being seemingly more complicated, the Schlesinger equations (25) have a Hamiltonian structure. The most direct way to illustrate it is to consider the EDO associated with the generic Fuchsian system (23). Let us choose a gauge such that

$$\text{Tr} A_\infty = \theta_\infty. \qquad (26)$$

Consider the off-diagonal term $A_{12}$ of (23). It is of the form

$$A_{12}(z) = \frac{k(z - \lambda)}{z(z - t)}, \qquad (27)$$

where $k, \lambda$ are linear functions of $(A_0)_{12}$ and $(A_t)_{12}$. Now, by writing the solution as

$$\Phi(z) = \begin{pmatrix} f_+^{(1)}(z) & f_+^{(2)}(z) \\ f_-^{(2)}(z) & f_-^{(2)}(z) \end{pmatrix}, \qquad (28)$$

one can check that the elements of the first row $f_+^{(1,2)}(z)$ satisfy

$$\frac{d^2}{dz^2} f_+^{(1,2)} + p(z)\frac{d}{dz} f_+^{(1,2)} + q(z)f_+^{(1,2)} = 0,$$
$$p(z) = \frac{1 - \theta_0}{z} + \frac{1 - \theta_t}{z - t} - \frac{1}{z - \lambda}, \qquad (29)$$
$$q(z) = -\frac{1}{4} + \frac{C_0}{z} + \frac{C_t}{z - t} + \frac{\mu}{z - \lambda},$$

where $C_0$, $C_t$, $\lambda$ and $\mu$ are complicated functions of the entries of $A(z)$. This EDO has, along with the singular points at $z = 0, t, \infty$, an extra singularity at $z = \lambda$. This singularity can be checked to be an apparent one: the solutions of the indicial equation at $z = \lambda$ gives $\alpha_\lambda^+ = 0, 2$ and there is no logarithm behavior due to the algebraic

relation between the parameters:

$$\mu^2 - \left[\frac{\theta_0 - 1}{\lambda} + \frac{\theta_t - 1}{\lambda - t}\right]\mu + \frac{C_0}{\lambda} + \frac{C_t}{\lambda - t} = \frac{1}{4}. \qquad (30)$$

This relation means that the change of $t$ has to be accompanied by a change in $\lambda$ and $\mu$ so that the relation is maintained.

The Schlesinger system (25), when written in these parameters, yield the Painlevé V equation for the following function of the entries of the $A_i$:

$$y(t) = \frac{(A_0)_{11}(A_t)_{12}}{(A_t)_{11}(A_0)_{12}} = \frac{\theta_0 + \theta_t - \theta_\infty - (2\mu - 1)(\lambda - t)}{\theta_0 + \theta_t - \theta_\infty - (2\mu - 1)\lambda}.$$

The Painlevé V is part of the family of second order differential equations with rational coefficients and the Painlevé property: all the branch points of the solutions are fixed, determined by the ODE itself [11, 18]. These equations define new special functions, and the Painlevé V in particular has been useful to compute correlation functions of strongly coupled Bosonic systems [19], distribution functions of random matrix theory, certain limits of conformal blocks and the $XY$ model – see [20] for a (not exaustive) list of applications. It has also been shown to give exact analytic expressions for the scattering of massless fields in black hole backgrounds [21, 22].

We follow [13] and define the tau-function:

$$\frac{d}{dt} \log \tau(t, \vec{\theta}) = -\frac{1}{2} \text{Tr} \sigma_3 A_t - \frac{1}{t} \text{Tr} A_0 A_t \qquad (31)$$

which satisfies a third order non-linear ODE – the so-called $\sigma$-form of the Painlevé equations and it is defined up to a multiplicative constant.

The tau-function has the direct interpretation of generating function for correlations in field-theoretic applications of the Painlevé transcendents. Asymptotic expressions for the tau-function have been derived in [10] and the (irregular) conformal block interpretation was given in [23, 24], and the relevant results are listed in the Appendix. In order to describe it we define the composite monodromy parameter

$$2\cos\pi\sigma = \text{Tr}(M_t M_0) = \text{Tr}(M_\infty^{-1})$$
$$= 2\cos\pi\theta_\infty + e^{\pi i \theta_\infty} s_1 s_2, \qquad (32)$$

then the monodromy data can be written as $\vec{\theta} = \{\theta_0, \theta_t, \theta_\infty, \sigma, s_i\}$. We will assume generic (*i.e.*, non-multiples of $\pi$) values for the monodromy data $\vec{\theta}$ so these expressions can be locally inverted.

In terms of (31), the existence of monodromy matrices for the Rabi Fuchsian system (9) amounts to the existence of a solution to the tau-function given the initial

conditions:

$$\frac{d}{dt} \log \tau(t, \vec{\theta}) \bigg|_{t=-4g^2} = \frac{E + g^2}{2} + \frac{\Delta^2}{4g^2}$$
$$\frac{d^2}{dt^2} \log \tau(t, \vec{\theta}) \bigg|_{t=-4g^2} = \frac{1}{t^2} \operatorname{Tr} A_0 A_t = \frac{\Delta^2}{16g^4}, \tag{33}$$

which is guaranteed on general grounds. Note that in the case of interest $\theta_0 = \theta_t = E + g^2$ and $\theta_\infty = 0$, these conditions can be inverted to yield the non-trivial monodromy data – the Stokes parameters in our application. It should be stressed that the equations above solve the Rabi model in an implicit but combinatorial sense: the formulae given in the Appendix give an asymptotic expansion for the Painlevé V tau-function near $t = 0$. The same special type of Painlevé V system as above was studied in a series of papers [25–27], where the invariants of the isomonodromy flow were calculated, and, in the last of the series, a Toda chain structure outlined.

## QUANTIZATION

The pair of equations (33) provides an implicit solution of the Rabi model in terms of the Painlevé V tau-function. The quantization condition for the energy can be solved in a similar form by making use of the composite monodromy parameter (32). We start by considering the asymptotics of the solution of the Fuchsian system (9) with the behavior near $z = \infty$ given by (16). Near the regular singular points $z = 0, t$ the solution will behave as:

$$\Phi(z) = \begin{cases} G_0 z^{\begin{pmatrix} \theta_0 & 0 \\ 0 & 0 \end{pmatrix}} (\mathbb{1} + \mathcal{O}(z)) C_0, & z \to 0 \\ G_t (z - t)^{\begin{pmatrix} \theta_t & 0 \\ 0 & 0 \end{pmatrix}} (\mathbb{1} + \mathcal{O}(z - t)) C_t, & z \to t, \end{cases}$$

where $G_i$ ($i = 0, t$) are the matrices diagonalizing $A_i$, and $C_i$ are called the connection matrices. The monodromy matrices are diagonalized by the connection matrices, that is,

$$M_i = C_i^{-1} e^{\pi i \theta_i} C_i. \tag{34}$$

Thus, the $C_i$ are defined up to right multiplication. Without loss of generality we can take $\det C_i = 1$. The matrix $C_0 C_t^{-1}$ can be seen to "connect" the natural solutions of the system (9) at $z = 0$ and $z = t$.

We can use the monodromy matrix to solve for the eigenvalue problem. The solution of (9) is required from physical grounds to be analytic on the whole plane – it will have an essential singularity at $z = \infty$, but analyticity at $z = 0, t$ ensures that the quantum state defined by the solution has finite expectation values for the relevant physical quantities (like the bosonic number operator). This condition is translated to our language by requir-

ing that the matrix $C_0 C_t^{-1}$ which connects the natural solutions at $z = 0$ and $z = t$ is diagonal: the analytic solution at $z = 0$ will also be analytic at $z = t$. In principle the connection could be "upper triangular": the second solution at $z = 0$, which diverges as $z^{\theta_0}$, could be connected to a superposition of the divergent and the regular solutions at $z = t$, but one can easily see that this does not happen: consider the determinant of the fundamental matrix, $\det \Phi$, which satisfies the equation

$$\frac{d}{dz} \det \Phi = \left( \frac{\theta_0}{z} + \frac{\theta_t}{z - t} \right) \det \Phi. \tag{35}$$

This equation yields $\det \Phi = z^{\theta_0} (z - t)^{\theta_t}$, and can be used to "normalize" the solutions, in the sense that now the fundamental matrix has unit determinant. One can convince oneself that this normalization does not change the connection matrices $C_i$, but now the two natural solutions at any particular singular points have a similar behavior: $(z - z_i)^{\pm \theta_i / 2}$. This parity, a $\mathbb{Z}_2$ parity, was fundamental in Braak's work. In [22], it was though associated to a time-reversal symmetry. For our application, this symmetry guarantees that the vanishing of one of the off-diagonal elements of $C_0 C_t^{-1}$ will imply the vanishing of the other off-diagonal term. Hence $C_0 C_t^{-1}$ will be diagonal.

Now, a diagonal $C_0 C_t^{-1}$ implies, for the composite monodromy parameter $\sigma$, defined in (32), that

$$\cos \pi \sigma = \cos \pi (\theta_0 + \theta_t). \tag{36}$$

Therefore, the regularity of the solution is expressed as a quantization condition:

$$\sigma_n = 2n + \theta_0 + \theta_t = 2(E + g^2 + n), \qquad n \in \mathbb{Z}. \tag{37}$$

Since $\sigma$ is given in terms of the Stokes parameters $s_{1,2}$, this condition can be fed into the solution (33) to yield the quantized values for the energy $E_n$. The completion of this task requires the knowledge of the expansion of the tau-function given in the Appendix.

## PERSPECTIVES

The methods described here are useful not only to show the existence of the monodromy matrices, and hence Yang-Baxter integrability, for the Rabi model but it also provides a solution for the eigenvalue problem in terms of the transcendental equation (33). Given that there is a combinatorial expansion of the Painlevé V tau-function in terms of irregular conformal blocks, one can then implement a numerical/symbolic computation to complete the task of finding the eigenvalues, using the expansion given in the Appendix and the quantization condition (37).

Another interesting direction would be to use the pro-

posed formalism to other similar systems. For instance, the extension to the model with broken parity introduced by Braak. This consists in adding a term of the form $\gamma\sigma^x$ to the Rabi Hamiltonian, which seems to be a simple extension and amenable through the methods described here.

### Formulae for the Painlevé V $\tau$-function

Here we lift the relevant formulae from [24]. In the following we consider the tau-function as defined in [10]:

$$\tau(t) = t^{((\theta_0-\theta_t)^2-\theta_\infty^2)/4}[\tilde{\tau}(t)]^{-1}. \quad (38)$$

The expansion for the tau-function is of the form:

$$\tilde{\tau}(t,\vec{\theta}) = \sum_{n\in\mathbb{Z}} C(\{\theta_i\},\sigma+n)s^n t^{(\sigma+n)^2}\mathcal{B}(\{\theta_i\},\sigma+n;t),$$
$$(39)$$

where the irregular conformal block $\mathcal{B}$ is given as a power series over the set of Young tableaux $\mathbb{Y}$:

$$\mathcal{B}(\{\theta_i\},\sigma;t) = e^{-\theta_t t}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}(\{\theta_i\},\sigma)t^{|\lambda|+|\mu|}, \quad (40)$$

$$\mathcal{B}_{\lambda,\mu} = \prod_{(i,j)\in\lambda}\frac{(\theta_\infty+\sigma+i-j)((\theta_t+\sigma+i-j)^2-\theta_0^2)}{h_\lambda^2(i,j)(\lambda_j'+\mu_i-i-j+1+2\sigma)}\times$$
$$\prod_{(i,j)\in\mu}\frac{(\theta_\infty-\sigma+i-j)((\theta_t-\sigma+i-j)^2-\theta_0^2)}{h_\mu^2(i,j)(\lambda_i+\mu_j'-i-j+1+2\sigma)}. \quad (41)$$

where $\lambda$ denotes a Young tableau, $\lambda_i$ is the number of boxes in row $i$, $\lambda_j'$ is the number of boxes in column $j$ and $h_\lambda(i,j) = \lambda_i + \lambda_j' - i - j + 1$ is the hook length related to the box $(i,j)\in\lambda$. The structure constants $C$ are rational products of Barnes functions:

$$C(\{\theta_i\},\sigma) = \prod_{\epsilon=\pm} G(1+\theta_\infty+\epsilon\sigma)G(1+\theta_t+\theta_0+\epsilon\sigma)\times$$
$$G(1+\theta_t-\theta_0+\epsilon\sigma)/G(1+2\epsilon\sigma), \quad (42)$$

where $G(z)$ is defined by the functional equation $G(1+z) = \Gamma(z)G(z)$. The parameters $\sigma$ and $s$ in (39) are re-

lated to the "constants of integration" of the Painlevé V equation. The $\sigma$ is the same monodromy parameter as (32), whereas $s$ has a rather lengthy expression in terms of monodromy data that can be read from [10]. The Painlevé V tau-function was also considered in great detail in [13]. The particular set of parameters considered here were also considered in [25–27].

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